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DIOPHANTINE APPROXIMATION IN R^n

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ABSTRACT. A modification of the Ford geometric approach to the problem of approximation of irrational real numbers by rational fractions is developed. It is applied to find an upper bound for the Hurwitz constant for a discrete group acting in a hyperbolic space. A generalized Khinchine's approximation theorem is also given.

1. INTRODUCTION

Let α be a real irrational number. In 1891 A. Hurwitz [14] showed that the inequality

$$|\alpha - ac^{-1}| < \frac{1}{k|c|^2} \tag{1}$$

has infinitely many solutions in coprime integers a and c when $k = \sqrt{5}$, and $\sqrt{5}$ is the best constant possible. The first geometric proof of this result was obtained by L. Ford in [12] where he makes use of properties of the modular group. The main purpose of this paper is to obtain similar results for higher-dimensional spaces using the Ford geometric approach as it is developed in [27].

We denote by V the n-dimensional Euclidean space $\mathbf{R}^n$. The upper half-space $H^{n+1} = \{(z, t) : z \in V, t > 0\}$ with the metric $ds^2 = t^{-2}(|dx|^2 + dt^2)$ can be used as a model of the $(n+1)$-dimensional hyperbolic space. Here $|\cdot|$ is Euclidean length in V. Let $\mathrm{Con}(n)$ denote the group orientation-preserving isometries of H^{n+1}. Let G be a geometrically finite discrete subgroup of $\mathrm{Con}(n)$ (see [3]). A geodesic in H^{n+1} is a semicircle or a ray which is orthogonal to V. An element $g \in \mathrm{Con}(n)$ extends to a conformal transformation of $V \cup H^{n+1}$, the closure of H^{n+1}. Hence, g will fix a point either in H^{n+1} or on its boundary V. The type of g is *elliptic, parabolic*, or *loxodromic* depending on whether it has a fixed point in H^{n+1}, a single fixed point in V, or exactly two fixed points in V (see e.g. [3]). If g is loxodromic, the geodesic connecting its fixed points is called the *axis of* g. The transformation g is *hyperbolic* if it is loxodromic and every plane containing its axis is g-invariant. We denote by $\mathscr{P}$ the set of parabolic fixed points (*cusps*) of G.

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Let P be a Dirichlet polygon of $G_\infty = \mathrm{Stab}(\infty, G)$ in V. Denote $P_\infty = \{(z, t) \in H^{n+1} : z \in P\}$. The region

$$D = P_\infty \cap \{x \in H^{n+1} : |g'(x)| < 1, \ g \in G\}$$

is an *isometric* fundamental domain for G in H^{n+1} (see [3] or [4]). Here $g'(x)$ stands for the Jacobian of the transformation g. In the sequel, we shall have to deal mainly with the components of the boundary of D of dimensions 0, 1, and n. We shall call them *vertices* (or *cusps*), *edges*, and *faces* of D respectively.

Let $\alpha \in V - \mathscr{P}$. Assume that $\infty \in \overline{D}$, where $\overline{D}$ is the closure of D. It is known (see e.g. [3] or [5]) that there is a constant $k > 0$, depending only on G, such that the inequality

$$|\alpha - g(\infty)| < r^2(g)/k \tag{2}$$

holds for infinitely many cosets of G_∞ in G. Here $r(g)$ is the radius of the *isometric sphere* $I(g) = \{x \in H^{n+1} : |g'(x)| = 1\}$ of $g \in G$. If $n = 1$ and G is the modular group, (2) is reduced to (1).

For a fixed $\alpha \in V - \mathscr{P}$, we denote by $k(\alpha)$ the supremum of all such k in (2). The set of numbers

$$\mathscr{L}(G) = \{1/k(\alpha), \alpha \in V - \mathscr{P}\}$$

is called the *Lagrange spectrum* for G and $C(G) = \sup \mathscr{L}(G)$ the *Hurwitz constant* for G.

Let D be an isometric fundamental domain of G. Let an edge σ of D lie on a geodesic L which is not a vertical ray. The point of L farthest from V is called the *summit* of σ.

Define k_G to be the largest value of k such that the connected parts of D lying below $t = k/2$ are pyramidal regions bounded by the faces of D which meet at a vertex or cusp of D and the Euclidean plane $t = k/2$ (see Figure 1). If the summit of every edge of D belongs to the closure of D then k_G is twice the distance from V to the summits of D.

Theorem 1. *Let G be a geometrically finite group acting in the $(n+1)$-dimensional hyperbolic space H^{n+1}. Let the n-dimensional Euclidean space V be the limit set and $\mathscr{P}$ the set of parabolic fixed points of G. Suppose that $\infty \in \mathscr{P}$. Let $\alpha \in V - \mathscr{P}$. Then there are infinitely many left cosets of $G_\infty = \mathrm{Stab}(\infty, G)$ in G whose members g satisfy (2) with $k = k_G$. Thus, $C(G) \leq 1/k_G$.*

Theorem 1 will be proved in §2. In §3, the unit ball model $\mathbf{B}$ for the hyperbolic space is introduced. Here we discuss the problem of approximation of a point on the unit sphere $\mathbf{S}$, the boundary of $\mathbf{B}$, by the orbit of some point in $\mathbf{B}$, and prove Theorem 4 which is an analog of Theorem 1 for the ball model of the hyperbolic space. The Vahlen group of Clifford matrices is introduced in §4. For the arithmetic subgroups of Vahlen's group the inequality (2), where α is a vector in an n-dimensional Euclidean space, can be written in form (1) for any $n > 0$ (Theorem 5). For these groups, we also give the generalized Khinchine's approximation theorem proved by Sullivan [23] (Theorem 6). The results obtained are applied to some triangular Fuchsian groups, including Hecke groups (Examples 1 and 2) and the arithmetic subgroups of Vahlen's group in §5 where the Hurwitz constants for three- and four-dimensional spaces are found. In particular, we find the approximation constant for the ring of Hurwitz integral

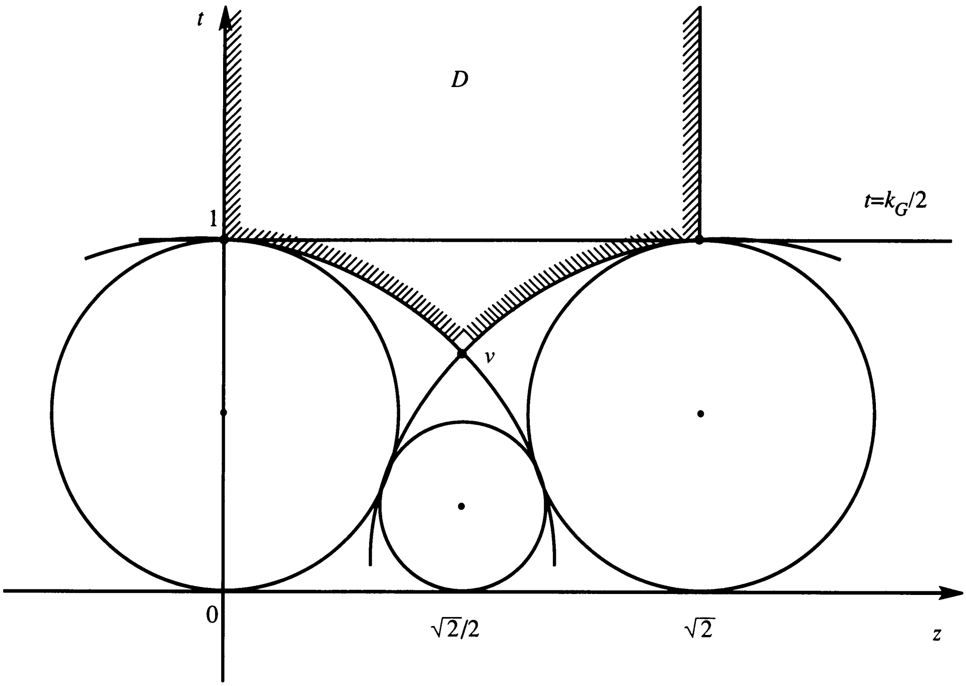

FIGURE 1

quaternions (Example 4). This result was first obtained by A. Speiser [22] and A. L. Schmidt [21].

For cofinite Fuchsian groups, the results of this paper are improved in [26].

The author thanks the referee for his useful remarks which led to an improvement of this work.

2. k-NEIGHBORHOODS OF VERTICES AND CUSPS

Here we shall prove Theorem 1. Our arguments are based on extensions and modifications of the method in [27] which we now describe.

Let $\alpha \in V - \mathscr{P}$. Denote by $L = L(\alpha)$ the vertical half-line in H^{n+1} through α. Let $g \in G$. For any $k > 0$, let $\mathscr{R}(g, k)$ be the open Euclidean ball in H^{n+1} tangent to V at $g(\infty)$ having radius r^2/k where $r = r(g)$ is the radius of the isometric sphere $I(g)$ of $g \in G$. We have $\mathscr{R}(g, k) = g(\mathscr{R}_k)$ where $\mathscr{R}_k = \mathscr{R}(\mathrm{id}, k) = \{(z, t) \in H^{n+1} : t \geq k/2\}$. We denote the boundaries of the horoballs $\mathscr{R}(g, k)$ and $\mathscr{R}_k$ by $Q(g, k)$ and Q_k respectively. Thus, the inequality (2) holds if and only if L cuts the horosphere $Q(g, k)$.

Since α is not a parabolic fixed point, the line L passes through infinitely many fundamental domains $g(D)$, $g \in G$. Let the fundamental domains through which L passes be $g_n(D)$, $n = 1, 2, \ldots$, taken in order as a point (z, t) moves along L from ∞ to α. Let (α, t_n) be the point of intersection of L with the common boundary of $g_{n-1}(D)$ and $g_n(D)$. Denote $L_n = \{(\alpha, t) \in L : 0 < t < t_n\}$. Let $\mathscr{N}(k)$ be the region in $H^{n+1} \cup \mathscr{P}$ which is exterior to all $\mathscr{R}(g, k)$, $g \in G$.

Let $\mathscr{C}$ be a cycle of vertices of D. The fact that $\mathscr{C}$ lies on some horosphere

$t = \text{const}$ can be shown as in the two-dimensional case (see, e.g. [4, p. 229 and p. 288]). Assume that $v = (z, t) \in \mathscr{C}$ and $t < k/2$. Since $k < k_G$, the component of the closure of D which contains v is a pyramidal region $\mathscr{A}(v)$. By definition, the region $\mathscr{N}(k)$ is covered by $g(\mathscr{A}(v))$, $g \in G$, where v runs through all the vertices and cusps of D.

Let B be a face of $\mathscr{A}(v)$ lying on the isometric sphere ϕ of some $h \in G$. Let λ be the part of the boundary of B that lies in $t = k/2$. The transformation h can be represented as a superposition of the inversion with respect to ϕ and the reflection in the vertical plane through the midpoint between v and $v' = h(v)$ and possibly the rotation around the vertical line through the center of the isometric sphere ϕ' of h^{-1} (see [3, p. 10]). Thus, if B' is a face of $\mathscr{A}(v')$ lying in ϕ', with λ' in $t = k/2$, then $\lambda' = h(\lambda)$ and $B' = h(B)$. It follows that the connected component of $\mathscr{N}(k)$ that contains v is covered by the images of $\mathscr{A}(v)$, $v \in \mathscr{C}$. We denote such a component of $\mathscr{N}(k)$ by $N(v, k)$. The boundary of $N(v, k)$ consists of parts of horospheres $Q(g, k)$, $g \in G$. Figure 1, where G is the Hecke group G_4 (see Example 1) and $k = k_G = 2$, shows the triangular region $\mathscr{A}(v)$ bounded by horocycle Q_2, $t = k/2 = 1$, and isometric circles $|x| = 1$ and $|x - \sqrt{2}| = 1$. Here $x = (z, t) \in H^2$. The region $N(v, k)$ is bounded by $t = 1$ and horocycles with bases at 0, $\sqrt{2}/2$, and $\sqrt{2}$.

If $v \in H^{n+1}$ then $N(v, k)$ is covered by a finite number of fundamental domains $g(D)$. If v is a cusp of D then $N(v, k)$ is not covered by a finite number of fundamental domains, but for any $\varepsilon > 0$ the set $\{u \in N(v, k) : |u - v| > \varepsilon\}$ is. (Here $|u - v|$ is the Euclidean distance between u and v.) Thus, the intersection of a geodesic in H^{n+1} with any $N(v, k)$ is covered by a finite number of fundamental domains $g(D)$, $g \in G$.

Assume now that Theorem 1 is false. Then there exists an integer n such that $L_n \subset \mathscr{N}(k_G)$. Hence L_n is contained in some connected component $N(v, k)$ of $\mathscr{N}(k_G)$. Since α is not a cusp, L_n cannot be covered by a finite number of fundamental domains $g(D)$, $g \in G$. The contradiction obtained proves the theorem.

3. The ball model

In this section, the unit ball $\mathbf{B} = \{x \in \mathbf{R}^{n+1} : |x| < 1\}$ with a metric derived from the differential $ds = 2(1 - |x|^2)^{-1}|dx|$ is used as a model for the $(n+1)$-dimensional hyperbolic space. The group of orientation-preserving isometries of $\mathbf{B}$ will be denoted by $\mathrm{Con}(\mathbf{B})$. Let G be a geometrically finite discrete subgroup of $\mathrm{Con}(\mathbf{B})$. The unit sphere $\mathbf{S}$, the boundary of $\mathbf{B}$, is the limit set of G. As in §1, we denote by D an isometric fundamental domain for G in $\mathbf{B}$. It coincides with the Dirichlet polygon for G with center 0 (see [4] or [3]). If $\mathrm{Stab}(0, G)$ is not trivial, D is the intersection of the fundamental domain of $\mathrm{Stab}(0, G)$ in $\mathbf{B}$ with the region $\{x \in \mathbf{B} : |g'(x)| < 1 \text{ for all } g \in G\}$.

Let $\alpha \in \mathbf{S} - \mathscr{P}$. (Now $\mathscr{P}$ can be empty.) We first discuss the approximation of α by elements of the orbit $G0$. It is known that there is a positive constant k, independent of α, such that the inequality

$$|\alpha - g(0)| < r^2(g)/k \tag{3}$$

holds for infinitely many $g \in G$. Application of the reflection with respect to the unit sphere $\mathbf{S}$ (see e.g. [4, p. 41]) shows that $|\alpha - g(\infty)| = |g(\infty)|\,|\alpha - g(0)|$ so that (3) is equivalent to (2). Let ϕ be the geodesic which contains an edge σ

of D. The point of ϕ closest to the origin is called the *summit* of σ. Define ρ_G to be the smallest value of ρ such that the connected components of D exterior to $|x| = \rho$, $x \in \mathbf{B}$, are the pyramidal regions bounded by the faces of D which meet at a vertex or a cusp of D and the sphere $K = \{x \in \mathbf{B} : |x| = \rho\}$. If the summit of every face of D belongs to the closure of D, then ρ_G is the distance from the origin to the farthest edge of D.

Replacing the horospheres $Q(g, k)$ from §2 by hyperbolic spheres $g(K)$, $g \in G$, and the vertical ray $L(\alpha)$ in H^{n+1} by the Euclidean radius $[0, \alpha)$ in $\mathbf{B}$, one can prove the following analog of Theorem 1 for the unit ball.

Lemma 2. *Suppose that G is a geometrically finite group acting in the unit $(n+1)$-ball $\mathbf{B}$. Let $\mathbf{S}$ be the unit n-sphere, the limit set of G, and $\mathscr{P}$ the set of parabolic fixed points of G. Let $\alpha \in \mathbf{S} - \mathscr{P}$. Then there are infinitely many $g \in G$ satisfying* (2) *where*

$$k = k_G = 2\frac{1 - \rho_G^2}{1 + \rho_G^2} = \frac{2}{\cosh R}.$$

Here R is the hyperbolic radius of the sphere $|x| = \rho_G$.

Let $w \in D$. Suppose that the orbit Gw is used to approximate $\alpha \in \mathbf{S} - \mathscr{P}$. Let $W(0) = w$ for some $W \in \mathrm{Con}(\mathbf{B})$. Notice that, up to rotation about the origin, W is defined uniquely by w. Denote $\alpha_1 = g^{-1}(\alpha)$, and $g_1 = W^{-1}gW$. Let r_1 and r_2 be the radii of the isometric spheres of g_1 and gW respectively.

Lemma 3. *The inequality*

$$|\alpha_1 - g_1(0)| < r_1^2/k$$

holds for infinitely many $g_1 \in G' = W^{-1}GW$ if and only if

$$|\alpha - g(w)| < r_2^2/k$$

does.

Proof. Since $g(w) \to \alpha$,

$$\frac{|\alpha_1 - g_1(0)|}{|\alpha - g(w)|} \to \frac{1 - |w|^2}{|\alpha - w|^2} = |(W^{-1})'(\alpha)| \tag{4}$$

(see e.g. [20, pp. 8–12]). It is easily seen that $r^2 = |g(\infty)|^2 - 1$, if $g(0) \neq 0$. Here r is the radius of the isometric sphere of $g \in \mathrm{Con}(\mathbf{B})$. Hence

$$r_2^2/r_1^2 = |(W^{-1})'(g(w^*))| \tag{5}$$

where $w^* = W(\infty)$ is the image of w under the reflection with respect to $\mathbf{S}$. Since $g(w^*) \to \alpha$, Lemma 3 follows from (4) and (5).

Lemma 3 implies that the problem of approximation of α by the orbit Gw is equivalent to the problem of approximation of α_1 by the orbit $G'0$ (or by $G'\infty$). Lemma 2 and 3 imply the following.

Theorem 4. *Suppose that G is a geometrically finite group acting in the unit $(n+1)$-ball $\mathbf{B}$. Let $\mathbf{S}$ be the unit n-sphere, the limit set of G, and $\mathscr{P}$ the set of parabolic fixed points of G. Let $w \in \mathbf{B}$ and $D(w)$ be the Dirichlet polyhedron with center at w. Let $\alpha \in \mathbf{S} - \mathscr{P}$. Then there are infinitely many $g \in G$ satisfying*

$$|\alpha - g(w)| < \frac{\cosh R}{2} r^2(gW)$$

where R *is the hyperbolic distance from* w *to a farthest edge of* $D(w)$ *and* $W \in \mathrm{Con}(\mathbf{B})$ *such that* $W(0) = w$.

4. Vahlen's group of Clifford matrices

In the present section we first review some basic facts on Clifford algebras, define Vahlen's groups of Clifford matrices, and introduce their $\mathbf{Q}$-arithmetic subgroups. The exposition is close to that from [25] (see also [8, 9, 10]). We refer the reader to [6] and [11] for details. For the discrete subgroups defined, we restate Theorem 1 and, applying Sullivan's results [23], give the generalized Khinchine's approximation theorem.

Let field K be either $\mathbf{R}$ or $\mathbf{Q}$. Let E be an $(n-1)$-dimensional vector space over K. Let $\Phi : E \times E \to K$ be a nondegenerated symmetric bilinear form with associated quadratic form $q(x) = \Phi(x, x)$. Let $a_q := x \otimes y + y \otimes x - 2\Phi(x, y)$ $(x, y \in E)$ be the two-sided ideal in the tensor algebra $T(E)$ of E. The Clifford algebra of q is defined as $\mathscr{C}(K, q) := T(E)/a_q$; for $n = 1$, $\mathscr{C}(K, q) = K$. We identify K and E with their canonical images in $\mathscr{C}(K, q)$ and define the n-dimensional vector space

$$V(K, q) := K \cdot 1 + E \subseteq \mathscr{C}(K, q).$$

Let $e_1, \dots, e_{n-1}$ be a basis of E orthogonal with respect to q. Then we have

$$e_k^2 = q(e_k), \quad e_k e_m = -e_k e_m \qquad (k, m = 1, \dots, n-1; k \neq m).$$

Let J_n be the set of subsets of $\{1, \dots, n\}$. For $M \in J_n$, $M = \{k_1, \dots, k_r\}$ with $k_1 < \cdots < k_r$, we define $e_M := e_{k_1} \cdot \cdots \cdot e_{k_r}$, $e_\varnothing := 1 \in \mathscr{C}(K, q)$. Then 2^{n-1} elements $\{e_M : M \in J_n\}$ constitute a basis of $\mathscr{C}(K, q)$.

An element of $\mathscr{C}(K, q)$

$$z = x_0 + x_1 e_1 + \cdots + x_{n-1} e_{n-1} \in V(K, q)$$

is called a *vector* and is identified with $(x_0, \dots, x_{n-1}) \in K^n$. Products of nonzero vectors form the *Clifford group* T_{n-1}.

There are three involutions defined on $\mathscr{C}(K, q)$:

(i) the *main involution*, $x \to x'$, obtained by replacing each e_m with $-e_m$, satisfying $(xy)' = x'y'$;

(ii) the *main anti-involution*, $x \to x^*$, obtained by reversing the order of the factors in each term $e_{k_1} \cdot \cdots \cdot e_{k_r}$, satisfying $(xy)^* = y^* x^*$;

(iii) the *conjugation*, $x \to \bar{x} = x'^* = x^{*\prime}$, satisfying $\overline{xy} = \bar{x}\bar{y}$.

We define the quadratic form $\widehat{Q} : \mathscr{C}(K, q) \to K$ on the vector space $\mathscr{C}(K, q)$ so that

$$x\bar{x} = \widehat{Q}(x) e_\varnothing + \sum_{|M|>0} \lambda_M e_M \tag{6}$$

for all $x \in \mathscr{C}(K, q)$ (cf. [8]). The restriction of $\widehat{Q}$ to $V(K, q)$ will be denoted by Q. For $x, y \in V(K, q)$, we have

$$x\bar{x} = Q(x), \qquad x\bar{y} + y\bar{x} = 2(x, y) \in \mathbf{R} \tag{7}$$

where $2(x, y) = Q(x+y) - Q(x) - Q(y)$. Elements of T_{n-1} satisfy

$$x\bar{x} = Q(x_1) \cdot \cdots \cdot Q(x_r) = \widehat{Q}(x),$$

where $x = x_1 \cdot \dots \cdot x_r$, $x_i \in V(K, q)$ $(i = 1, \dots, r)$. It follows that, for any $x \in T_{n-1}$,

$$x^{-1} = \bar{x}/\widehat{Q}(x).$$

Since $e_1, \dots, e_{n-1}$ is an orthogonal basis of E with respect to q,

$$q(x) = q_d := d_1x_1^2 + \dots + d_{n-1}x_{n-1}^2$$

for $x \in E$ $(d_k = q(e_k),\ k = 1, \dots, n-1)$. Let

$$\mathscr{C}_q := \mathscr{C}(\mathbf{Q}, q), \quad V_q := V(\mathbf{Q}, q). \tag{8}$$

Let

$$q_0(x) := -x_1^2 - \dots - x_{n-1}^2.$$

Denote $\mathscr{C} := \mathscr{C}(\mathbf{R}, q_0)$ and $V := V(\mathbf{R}, q_0)$. For $x \in \sum_{M \in J_n} \lambda_M e_M \in \mathscr{C}$,

$$|x|^2 := \widehat{Q}(x) = \sum_{M \in J_n} \lambda_M^2$$

where the quadratic form $\widehat{Q}(x)$ is defined by (6) and $|x|$ denotes the Euclidean norm of x. If $x \in V$, then, from (7), $|x|^2 = x\bar{x} = \bar{x}x$.

The upper half-space

$$H^{n+1} := \{z + te_n : z \in V,\ t > 0\}$$

is a model of $(n+1)$-dimensional hyperbolic space. Vahlen's group of projective Clifford matrices $PSV_{n-1} := SV_{n-1}/\{\pm 1\}$ (see [1, 2, 8, 9, 17, 24]) where

$$\begin{aligned} SV_{n-1} := \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathscr{C}) : a, b, c, d \in T_{n-1} \cup \{0\}, \\ a\bar{b},\ \bar{c}d \in V,\ ad^* - bc^* = 1 \Big\} \end{aligned} \tag{9}$$

acts on H^{n+1} by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x := (ax + b)(cx + d)^{-1} \tag{10}$$

preserving the Poincaré metric (see e.g. [2] or [9]). It is isomorphic to the group of orientation-preserving hyperbolic isometries. Clearly, $SV_0 = SL_2(\mathbf{R})$, and, when $q(x) = -x^2$, $SV_1 = SL_2(\mathbf{C})$.

It follows from (9) that, for any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SV_{n-1}$, we have $a\bar{c}, b\bar{d} \in V$ (see, e.g. [8, p. 376]) and, if $c \neq 0$, then, by (10)

$$g(\infty) = ac^{-1}, \quad g^{-1}(\infty) = -c^{-1}d. \tag{11}$$

The set $|cz + d| = 1$ in V is the isometric sphere $I(g)$ with radius $|c|^{-1}$ and center $g^{-1}(\infty) = -c^{-1}d$. Similarly, $I(g^{-1})$ has the same radius and the center $g(\infty) = ac^{-1}$ (see [2]).

An arithmetic subgroup of SV_{n-1} is defined as follows [8]: A subring $\mathscr{T}$ of a $\mathbf{Q}$-algebra A (with unity element) is called a ***Z**-order* if $\mathscr{T}$ has the same unity element as A and the additive group of $\mathscr{T}$ is finitely generated and contains a **Q**-basis of A. A **Z**-order $\mathscr{T} \in \mathscr{C}_q$ (see (8)) is called *compatible* if it is stable under the involutions $^-$ and $'$ of $\mathscr{C}_q$. For a compatible **Z**-order $\mathscr{T}$, let

$$
\begin{aligned}
\Lambda &:= V \cap \mathscr{T}, \\
SV(\mathscr{C}_q) &:= SV_{n-1} \cap M_2(\mathscr{C}_q), \\
G = SV(\mathscr{T}) &:= SV(\mathscr{C}_q) \cap M_2(\mathscr{T}).
\end{aligned} \tag{12}
$$

The group G is a discrete subgroup in SV_{n-1}. If $n = 1$, $\mathscr{C}_q = \mathbf{Q}$, and $\mathscr{T} = \mathbf{Z}$, $SV_0(\mathbf{Z}) = SL_2(\mathbf{Z})$. For $n = 2$, $\mathscr{C}_q = \mathbf{Q}(\sqrt{-d})$ is an imaginary quadratic field, and $\mathscr{T}$ is some order in $\mathbf{Q}(\sqrt{-d})$.

If q is negative definite, the group G acts discontinuously on H^{n+1}, and the volume of the quotient H^{n+1}/G is finite (see [9, p. 262]). Let P be a fundamental parallelogram for the lattice $\Lambda \in V$. The region $P_\infty = \{(z, t) \in H^{n+1} : z \in P, t > 0\}$ is a fundamental region of $\operatorname{Stab}(\infty, G)$ in H^{n+1}. The intersection of P_∞ with the region in H^{n+1} satisfying the inequalities

$$|cz + d|^2 + |c|^2 t^2 > 1 \tag{13}$$

for all $(c\ d) = (0\ 1)g$, $g \in G$, is an isometric fundamental domain D for G with a finite number of faces (see e.g. [3, 9]).

By (11), for the group G, the inequalities (2) can be rewritten in the form (1). Thus, we have the following.

Theorem 5. *Let $n > 0$. Let $q : E \to \mathbf{Q}$ be a negative definite quadratic form. Suppose that $\mathscr{T}$ is a compatible **Z**-order in Clifford algebra $\mathscr{C}_q$ and G is defined by* (12). *Let $\mathscr{P}$ be the set of parabolic fixed points of G. Let $\alpha \in V - \mathscr{P}$. Then there are infinitely many left cosets of* $\operatorname{Stab}(\infty, G)$ *in G whose members g satisfy*

$$|\alpha - ac^{-1}| = |a - g(\infty)| < \frac{1}{k_G|c|^2}$$

$(c \neq 0)$. *Thus,* $C(G) \leq 1/k_G$.

A similar result can be obtained for the ball model (cf. [8, p. 383]). Theorem 5 shows that, as in the case of $n = 1, 2$, and 4, when $V = \mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$ respectively, for any $n > 0$, an "irrational" vector in the n-dimensional space V can be approximated by "rational fractions" $g(\infty) = ac^{-1} \in V$, $g \in G$, whose numerators a and denominators c are from the Clifford group: $a, c \in T_{n-1} \cap \mathscr{T}$. (Here **H** stands for the set of the Hamilton quaternions.)

The results obtained by Sullivan in [23, §§4–6], when applied to the group G, defined by (12), give the following generalized Khinchine's approximation theorem.

Theorem 6. *For almost all α in n-dimensional Euclidean space, $n > 0$, there are infinitely many left cosets of* $\operatorname{Stab}(\infty, G)$ *in the group G defined by* (12), *so that*

$$|\alpha - ac^{-1}| < \frac{f(|c|)}{|c|^2}, \quad ac^{-1} = g(\infty), \qquad g \in G,$$

if and only if

$$\int^\infty \frac{f^n(x)}{x}\, dx = \infty.$$

Remark. In [23, §7], a particular case of Theorem 6, when $n = 2$ and G is a Bianchi group, is considered. But the proof can be easily adapted to the general case.

5. Applications

Here we apply Theorems 1 and 3 to some of the arithmetic subgroups of the Vahlen groups.

Suppose that an edge σ of the fundamental domain D of the group G in Theorem 1 lies on the axis L of a loxodromic element $h \in G$. Let η and η' be the fixed points of h. Assume that σ is a *critical* edge of D, i.e. $k_G = |\eta' - \eta|$. As in [13], it can be shown that

$$k(\eta) = \sup |h(\eta') - h(\eta)|$$

where the supremum is taken over all $h \in G$. Hence the Hurwitz constant $C(G) \geq 1/k(\eta)$. A geodesic with endpoints η and η' is called *extremal* if $|\eta' - \eta| = \sup |h(\eta') - h(\eta)|$ where the supremum is taken over all $h \in G$. Applying Theorem 1, we obtain the following.

Lemma 7. *Let σ be an edge of the fundamental domain D for the group G such that $k_G = |\eta' - \eta|$ where η and η' are the endpoints of the geodesic L that contains σ. If L is an extremal geodesic, then the Hurwitz constant for the group G*

$$C(G) = \frac{1}{k_G} = \frac{1}{|\eta' - \eta|}.$$

The following statement will be used in Examples 1 and 2 below.

Corollary 8. *Let $n = 1$. Assume that the endpoints of the side σ of D from Lemma 7 are elliptic fixed points of G of an even order. Then $C(G) = 1/k_G$.*

Proof. Let L be the isometric circle of $h \in G$. Let v, v' be the endpoints of σ. Assume that R and R' generate $\mathrm{Stab}(v, G)$ and $\mathrm{Stab}(v', G)$ respectively. Let $R^{2n} = R'^{2m} = 1$. Then σ is a fundamental domain on L of the dihedral group $\langle R^n, R'^m \rangle$. No image of L can cut the interior of D. If there is another side σ' of D such that $\sigma' = g\sigma$, $g \in G$, then $g = h^{-1}$ since D is a convex polygon whose sides are conjugate in pairs (see [4]). Hence $k(\eta) = |\eta' - \eta|$ and L is extremal. By Lemma 7, $C(G) = 1/k_G$.

Example 1. Let

$$G_q = \left\langle \begin{pmatrix} 1 & 2\cos(\pi/q) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

where integer $q \geq 3$. It is a triangular group with signature $(0 : 2, q, \infty)$ known as a Hecke group. A fundamental domain of this group is bounded by the unit circle $|x| = 1$, $x = (z, t)$, and two vertical lines $z = \pm\cos(\pi/q)$. Thus, by Theorem 1, $k_G = 2$, and the Hurwitz constant $C(G_q) \geq 1/2$. If q is even, then, by Corollary 8, $C(G_q) = 1/2$. It is a result of Lehner [15]. The values of $C(G_q)$ for odd q's are found in [13], [16], and [26].

Example 2. Now let $\alpha = \pi/m$ and $\beta = \pi/l$ where m and l are positive integers such that $1/m + 1/l < 1/2$. Let $\rho = (\cos^2\beta - \sin^2\alpha)^{1/2}$. Let $x = z + it$.

Then (see [19, p. 87–88]) the group

$$G = \left\langle \frac{i}{\sin\alpha}\begin{pmatrix}\cos\beta & \rho \\ -\rho & -\cos\beta\end{pmatrix}, \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha}\end{pmatrix}\right\rangle$$

is a triangular group with signature $(0:2, l, m)$ which maps the unit disc $\mathbf{B}$ onto itself. Let $w = 0$. A Dirichlet polygon $D(0)$ of this group is bounded by the straight Euclidean lines joining the origin 0 with $P = \cos(\alpha+\beta)e^{i\alpha}/\rho$ and $\overline{P}$, and the circle $|x - (\cos\beta)/\rho| = (\sin\alpha)/\rho$. Hence $\rho_G = (\cos\beta - \sin\alpha)/\rho$ and, by Lemma 2, $k_G = 2(\sin\alpha)/\cos\beta$. If l is even then, by Corollary 8, the Hurwitz constant for G equals $(\cos\beta)/(2\sin\alpha)$. For q odd, the values of $C(G)$ are found in [26]. Similar results can be obtained for arbitrary triangular groups using their matrix representation given in [19, p. 105].

Example 3. Let $q = q_0$ and $\mathscr{T} = \mathbf{Z}^N$ where $N = 2^{n-1}$. Then $\Lambda = \mathbf{Z}^n$. If $n = 1, 2, 3$, or 4, then the inequalities (13) are reduced to

$$|z - m|^2 + t^2 > 1, \qquad m \in \Lambda. \tag{14}$$

A typical deep hole (see [7]) of Λ is $s = (\frac{1}{2}, \dots, \frac{1}{2})$ and the fundamental domain D for group G has a vertex v which lies above s. A typical critical edge σ of D passes through vertices v and v' where v' lies above the deep hole $s' = (-\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$, so that the summit of σ is above $u = (0, \frac{1}{2}, \dots, \frac{1}{2})$ (see Figure 2).

Thus, the diameter of the geodesic L which contains σ is

$$k_G = 2(1 - |u|^2)^{1/2} = (5-n)^{1/2}, \quad n = 1, 2, 3, \text{ or } 4.$$

By Theorem 1, the Hurwitz constant

$$C(G) \le (5-n)^{-1/2}. \tag{15}$$

The cases $n = 1$, when G is the modular group G_3, and $n = 2$, when G is the Picard group, are discussed in Example 1 and [27]. It is shown that the equality in (15) holds for $n = 2$ and does not hold for $n = 1$.

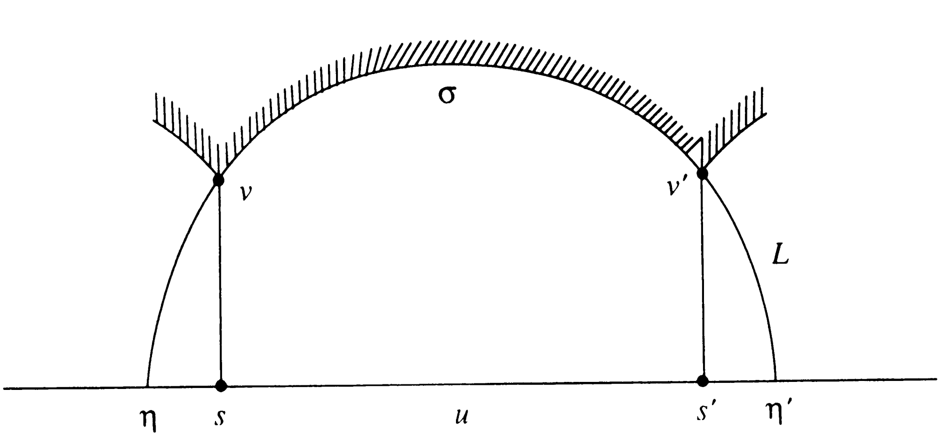

FIGURE 2

Let $n = 3$. We have $k_G = \sqrt{2}$ and the endpoints of the geodesic L,

$$\pm\frac{1}{\sqrt{2}} + \frac{1}{2}(i+j),$$

are the fixed points of the hyperbolic element

$$\begin{pmatrix} 3+2i+2j & 4 \\ 4 & 3-2i-2j \end{pmatrix} \in G$$

(see [2] or [18]). One can easily verify that L is extremal. Thus, by Lemma 7, $C(G) = 1/\sqrt{2}$.

When $n = 4$, the deep hole $s = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ of Λ is a cusp of D. We have $k_G = 1$, and Lemma 7 is not applicable.

Example 4. The group G in Example 3 can be extended by the group of symmetries of the fundamental domain D of G. For $n = 1, 2, 3$, this group coincides with the group of symmetries W of the unit cube P which is the fundamental domain of $\mathrm{Stab}(\infty, G)$ in the n-dimensional Euclidean space V. (Notice that, for $n = 3$, the extended group contains the Hurwitz group of integral quaternions.) In these cases, the Hurwitz constant of G is equal to the Hurwitz constant of the extended group.

When $n = 4$, as it is mentioned in Example 3, D has cusps at infinity and at $u = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$. Now D has an additional symmetry, reflection τ with respect to the sphere with center at u and radius one, which identifies these two cusps. Denote by G' the extension of G by both W and the reflection τ. The subgroup of translations Λ of $\mathrm{Stab}(\infty, G')$ coincides now with the group D_4 (see [7]) generated by 1, i, j, and $\zeta = \frac{1}{2}(1+i+j+k)$. The fundamental domain D' of G' has only one cusp at infinity and it is the intersection of the fundamental domain P_∞ of $\mathrm{Stab}(\infty, G')$ in H^5 and the region satisfying the inequalities (14) where $\Lambda = D_4$. A typical deep hole of D_4 is $s = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ and D' has a vertex v which lies above s. A typical critical edge σ of D' has endpoints v and v', where v' lies above the deep hole $s' = (\frac{1}{2}, 0, \frac{1}{2}, 0)$. We have $|s - s'| = 1/\sqrt{2}$, and since v lies on the unit sphere $|z|^2 + t^2 = 1$, $v = (s, 1/\sqrt{2})$, and $v' = (s', 1/\sqrt{2})$ (see Figure 2). It follows that the diameter of the geodesic L which contains σ is $k_{G'} = \sqrt{5/2}$ and the endpoints of L,

$$\tfrac{1}{4}(2+i+j) \pm \tfrac{1}{4}(i-j)\sqrt{5},$$

are the fixed poins of the hyperbolic element

$$\begin{pmatrix} 9-4(i-j-ij) & -8(i-j) \\ 8(i-j) & 9+4(i-j+ij) \end{pmatrix} \in G$$

(see [2] or [18]). One can verify that L is extremal. Hence, by Lemma 7, the Hurwitz constant of G' equals $(2/5)^{1/2}$. Since the lattice D_4 can be identified with the ring of Hurwitz integral quaternions $\mathscr{H}$, we have found the approximation constant for $\mathscr{H}$. This result was first obtained by A. Speiser [22] and A. L. Schmidt [21].

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